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**THE p- AND h-p VERSIONS OF THE FINITE ELEMENT METHOD
AN OVERVIEW**

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The p- and h-p versions of the Finite Element Method

An Overview

by

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Abstract

We survey the advances in the p- and the h-p versions of the finite element method. An up-to-date list of references related to these methods is provided.

1. Introduction and brief history.

The origins of the finite element method (FEM), like those of the spectral method, may be traced back a long time. If we understand the FEM as the application of variational principles and approximation by piecewise smooth functions, then this idea was already used by Leibnitz in 1696 (in one dimension with piecewise linear functions). In two dimensions, Schellbach [1] used triangulation and piecewise linear functions, (see also [2]). Nevertheless, the modern FEM era starts with the paper [3] which demonstrated the potential for the use of the computer. Since then, more than 30,000 papers have appeared (see [4], [5], [6]). These papers are generally based on the h-version of the FEM, where the accuracy of the approximate solution is achieved by refining the mesh while using low order polynomials on the mesh. (KR)

The spectral method, when understood to be the use of variational principles (or other methods, such as collocation), combined with the use of polynomials of high degree, was already known to Ritz. A method of this type was developed (among others) by Galerkin [7] and was discussed in detail in [8], for eg.. A number of further developments of this method, attributed to S.G. Michlin, may be found in his many papers and books. For example, in [9] he discussed principles for the selection of basis functions and outlined a program (based on polynomial approximation) for the Soviet computer M-20. For various theoretical aspects we refer to the important paper [10].

Both methods mentioned were found to have their strengths and weaknesses. The FEM provided considerable flexibility and was well suited for computer implementation. The spectral method offered high rates of convergence when the solution was smooth.

In the 1970s, B.A. Szabo, recognizing these aspects, suggested and implemented a combination of the two approaches to utilize the advantages of each. Today, this combination is called the p- and the h-p version of the

finite element method. If the mesh is fixed and the accuracy of the solution is achieved only by increasing the degree of the elements, we obtain the p-version of the FEM. (If the domain is a square or triangle, and is understood to be one element, then the p-version is identical to the Ritz method described, for example, in [8]). If we simultaneously refine the mesh and increase the degrees of elements uniformly or selectively, we obtain the h-p version.

The first theoretical paper addressing the p-version was [11] and the h-p version [12]. Since Szabo's original work, significant progress has been made for these methods in terms of theory, implementation and engineering applications. Some of these achievements are addressed in this paper.

The spectral·method has been applied extensively in the last 15 years to problems in fluid mechanics. Recently, there has been interest shown in using this method over partitioned domains (rather than a single one, see for eg. [13]). In this context, the spectral method over a partitioned domain is very similar to the h-p version, though the emphasis of the two methods is different - the h-p version of the FEM concentrating on the special needs of structural mechanics analysis, while the spectral method being specialized more for fluid mechanics.

There are many programs based on the h-version of the FEM, some major commercial ones being MCNASTRAN, ADINA, ANSYS and others. There are only two commercial programs based on the p- and h-p version, FIESTA and PROBE, in addition to a large research program called STRIPE. Other commercial programs based on the p-version are being developed at various places and will be on the market in the near future. The authors of this paper have experience with PROBE and references to it (rather than any alternative) are for convenience only.

The h-p version of the finite element method has various features which are reflected in the implementation and architecture of the program and are different from the h-version. Recent advances in computer hardware bring to the forefront the problem of reliability of computations and the repercussions of the rapidly changing ratio between computer and human costs. The h-p version offers various essentially new possibilities which the h-version does not. As examples, we mention the (a-posteriori) assessment of the errors of the FEM calculations, new possibilities in the modeling of plates and shells (with models of Reissner-Mindlin type being naturally created) and inherent parallelization. In addition, the h-p version shows remarkable robustness, for eg. with respect to locking phenomena.

In this paper we present a survey of the state of the art of the p and h-p versions. The emphasis is on the theoretical aspects related to their use in approximating elliptic equations stemming from structural mechanics.

2. The model problem.

Problems in structural mechanics and the mechanics of solids are typically characterized by elliptic partial differential equations with piecewise analytic data, pertaining to the boundary, boundary conditions, coefficients and right hand side. Consequently, one can expect special features in the solution which should be somehow exploited by the numerical method used. In this section, we mention some typical available results. For simplicity and brevity, we restrict our discussion to the two dimensional case.

Let $\Omega_j \subset R^2$, $j = 1,2,\cdots,M$ be simply connected domains with boundaries $\partial\Omega_j = \Gamma^{(j)} = \bigcup_{i=1}^{m_j} \bar{\Gamma}_i^{(j)}$. $\bar{\Gamma}_i^{(j)}$ are analytic simple arcs which we call edges, while the ends $A_i^{(j)}$, $A_{i+1}^{(j)}$ of the edges are called vertices. If $m_j = 3$,

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respectively $m_j = 4$, we call Ω_j a _curvilinear triangle_, respectively _quadrilateral_. Otherwise it will be a curvilinear polygon. We will assume that $\bar{\Omega}_j \cap \bar{\Omega}_i$ is either empty, is a common vertex, or is a common edge. Let Ω be the interior of $\bigcup_{j=1}^{M} \bar{\Omega}_j$ and assume that Ω is a domain. By Γ^Ω we denote the boundary of Ω. The edges $\Gamma_i^{(j)}$ not belonging to Γ^Ω will be called _interface edges._

We now consider a model problem for second order scalar elliptic differential equations written in the weak form. Let

$$(2.1) \qquad B(u,v) = \int_\Omega \left[\sum_{i,j=1,2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \right] dx_1 dx_2$$

where $a_{ij} = a_{ji}$ are analytic functions on $\bar{\Omega}_\ell$ satisfying the standard ellipticity condition

$$(2.2) \qquad \sum_{i,j=1,2} a_{ij} \xi_i \xi_j \ge \gamma_\ell (\xi_1^2 + \xi_2^2), \quad \gamma_\ell > 0, \quad \ell = 1, \cdots, M.$$

Further, let

$$(2.3) \qquad F_1(v) = \int_\Omega fv \, dx_1 dx_2$$

where f is an analytic function on $\bar{\Omega}_\ell$, $\ell = 1, \cdots, M$. Let ω be continuous on $D = \bigcup_{\Gamma_i \in D} \bar{\Gamma}_i$, $\bar{\Gamma}_i = \bar{\Gamma}_j^{(\ell)} \subset \Gamma^\Omega$ and analytic on $\bar{\Gamma}_i$. D will be called the Dirichlet boundary.

Finally, let $\mathcal{K} = \Gamma^\Omega - D$ be the Neumann boundary and let g be defined on $\mathcal{K}$ and analytic on every $\Gamma_i \in \mathcal{K}$, with

$$(2.4) \qquad F_2(v) = \sum_{\Gamma_i \in \mathcal{K}} \int_{\Gamma_i} g \, v \, ds.$$

The exact solution of our model problem is defined in the usual way: find $u_0 \in H^1(\Omega)$, $u_0 = \omega$ on D such that

(2.5)
$$B(u_0,v) = (F_1 + F_2)(v)$$

for all $v \in {}^0H^1 = \{v \in H^1(\Omega), v = 0 \text{ on } D\}$. If D is empty, then the usual solvability condition for $F_1 + F_2$ has to be satisfied. By $\|u\|_E = (B(u,u))^{1/2}$ we denote the energy norm of u. It is equivalent to the $H^1(\Omega)$ seminorm (and to the $H^1(\Omega)$ norm on ${}^0H^1(\Omega)$). The most important cases are when a_{ij} are constant on every Ω_ℓ, $\ell = 1, \cdots, M$.

A similar formulation holds for the elasticity problem. Here, $\underline{u} = (u_1, u_2)$ and

(2.6)
$$B(\underline{u}, \underline{v}) = \int_\Omega \left[\sum_{i,j,k,l=1,2} b_{ijkl} \varepsilon_{ij}(\underline{u}) \varepsilon_{kl}(\underline{v}) \right) dx_1 dx_2$$

$$\varepsilon_{ij}(\underline{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

with the standard assumptions about b_{ij} and the functional F. The assumptions about the piecewise analyticity are analogously formulated as before. Here the boundary conditions can be more general, combining traction components and displacements on particular Γ_j.

In the case of an isotropic material, the bilinear form is

(2.7)
$$B(\underline{u}, \underline{v}) = \frac{E}{1+\nu} \int_\Omega \left[\sum_{i,j=1,2} \varepsilon_{ij}(\underline{u}) \varepsilon_{ij}(\underline{v}) + \frac{\nu}{1-2\nu} (\text{div } \underline{u} \text{ div } \underline{v}) \right] dx_1 dx_2.$$

Here, ν is the Poisson ratio, $0 < \nu < \frac{1}{2}$ and E is the Young's modulus of elasticity. The form degenerates for $\nu \to \frac{1}{2}$ but regularity of the solution is preserved. A similar degeneration with preservation of the regularity properties also occurs for general anisotropic materials (see [14], [15], for eg.).

The behavior of the solution is essentially very similar in both the scalar case and the elasticity problem. The solution u_0

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a) is analytic in $\bar{\Omega}_j - \bigcup_{i,j} A_i^{(j)}$, $i = 1, \cdots, m_j$, $j = 1, \cdots, M$.

b) has a special singular behavior in the neighborhood of every $A_i^{(j)}$.

The behavior of the solution is best understood when the operator in our problem is the Laplacian. In this case, near a vertex, we have

$$(2.8) \qquad u_0 = \sum_{j=1}^{J} \sum_{s=0}^{S} \sum_{m=0}^{N} c_{jsm} \, \psi_{jsm}(\theta) r^{\alpha_j + m} \log^s r + \bar{u}_0.$$

Here, (r,θ) are local polar coordinates at the corner point under consideration. The decomposition (2.8) is such that the remainder $\bar{u}_0$ is smoother than the terms in the sum. The functions ψ_{jsm} are smooth (in our case piecewise analytic). We have $S = 0$ except for special cases, when $S = 1$ is possible. N may be 0 or positive. For eg., $N \neq 0$ in the cases when a_{ij} are nonconstant or $\Gamma_j^{(i)}$ are curved. For details we refer to [16]. The norm of $\bar{u}_0$ in (2.8) depends on the geometry and diverges to ∞ when the geometry converges to certain exceptional cases. The coefficients c_{jsm} are related to the stress intensity factors. They can be global, depending on u_0, or local, depending only on the input data at the vertex.

In the case of the elasticity equations, the results are similar, although not as detailed. The coefficients α_j in (2.8) can now be complex and the conditions for S and M are not completely characterized.

There is often a practical need in actual problems to know the values of α_j and the functions ψ_{jsm}. In [17] a general, adaptive, completely robust algorithm for determining these coefficients and functions is given. As an example of the complicated structure of these coefficients, we show the two-material (anisotropic) case when zero displacements are prescribed at the boundary (Fig.2.1).
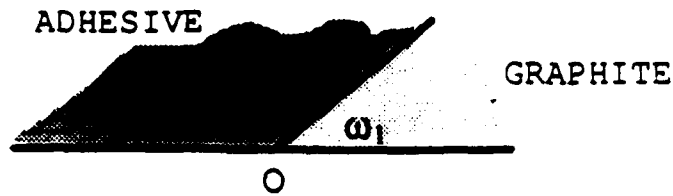
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ADHESIVE

GRAPHITE

ω_1

O

Fig.2.1 the scheme of the two material domain.

The two materials are typical anisotropic ones used in engineering. Graphite is highly anisotropic while adhesive is only slightly anisotropic.

Figs. 2.2 show the first six (or seven) α_i with smallest real part as functions of the angle (the accuracy is 10^{-5}). Fig. 2a shows the real part of α_j. If α_j is complex, values are denoted by circles. Fig. 2b depicts the imaginary part. For details see [17].
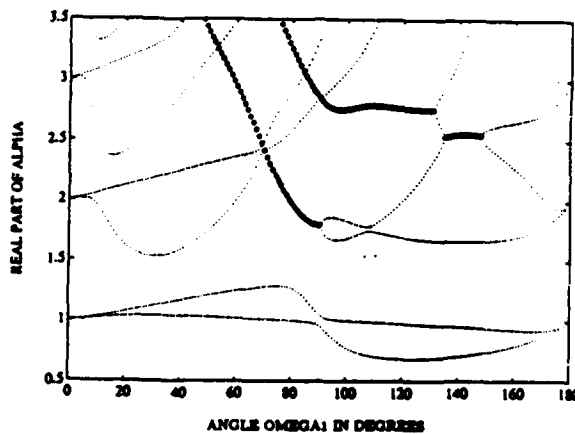
Fig.2.2a)   The real part of α_j.   Fig.2.2b) The imaginary part of α_j.
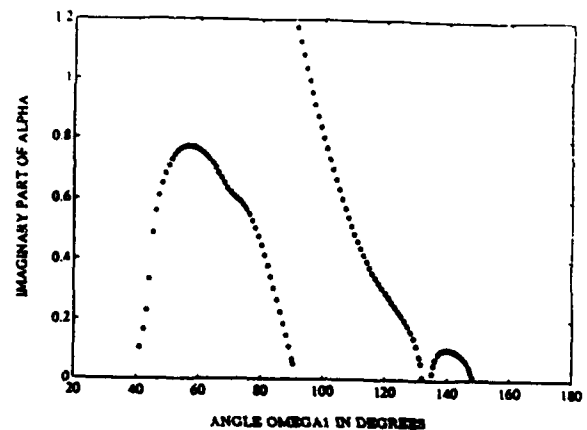
There is a vast literature devoted to the analysis of the decomposition (2.8). We mention here [18], [19], [20], [21] and references given therein. The above decomposition is valid when the input data are smooth but not necessarily analytic. Another characterization of the regularity, given in terms of countably normed spaces, was analyzed in [22], [23], [24], [25]. For

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example, it is shown that for the exact solution u_0,

$$(2.9) \qquad |D^\alpha u_0| \le C\left[r^{|\alpha|+\beta-1}\right]^{-1} d^{|\alpha|}\alpha! \quad 0 < \beta < 1$$

holds for all $\alpha \ge 1$. Here C and d are independent of α. In the above references, the complete characterization of these normed spaces together with corresponding trace spaces and extensions is analyzed.

In 3 dimensions, the situation is more complicated due to the presence of both edge and vertex singularities. As a result, the regularity may be different along different directions, leading to the use of anisotropic spaces.

Let us finally point out that the regularity of the solution of our model problem may also be characterized in terms of standard Sobolev (or Besov) spaces. Accordingly, if α is the minimum over all vertices A_j of the exponents α_j (or $Re(\alpha_j)$ if α_j are complex) in (2.8), then we have

$$(2.10) \qquad u \in H^k(\Omega) \ \forall \ k < 1 + \alpha.$$

Most classical finite element error estimates rely on regularity results of the form (2.10).

3. __The h-, p- and h-p version of the finite element method.__

To illustrate the basic results, we restrict ourself to very special cases, although the available results are completely general.

Let us consider the case when Ω is an L-shaped domain, as shown in Fig. 3.1.
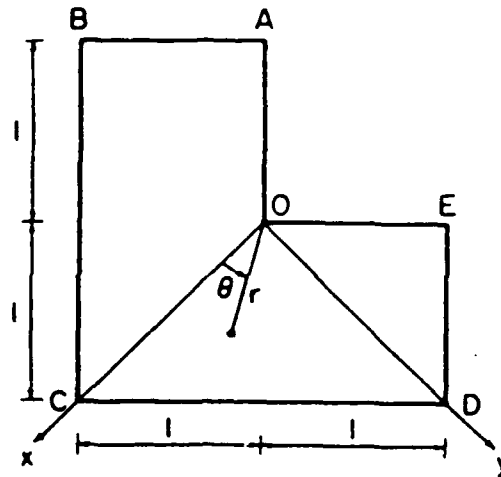
Fig.3.1 The scheme of the L-shaped domain.

We consider the elasticity problem (2.7) with $f = 0$ and traction boundary conditions such that

$$(3.1) \qquad u_1 = \frac{1}{2G} r^\lambda [(\kappa - Q(\lambda+1)) \cos \lambda\theta - \lambda \cos (\lambda-2)\theta]$$

$$u_2 = \frac{1}{2G} r^\lambda [(\kappa + Q(\lambda+1)) \sin \lambda\theta + \lambda \sin (\lambda-2)\theta]$$

where

$$\kappa = 3 - 4\nu,$$

$$G = E/2(1+\nu)$$

$$Q = - \frac{\lambda \sin (\lambda-1)\omega/2}{\sin (\lambda+1)\omega/2}$$

with $\nu = 0.3$, $\lambda = 0.544484$, $\omega = \frac{3}{2} \pi$. The sides OE, and OA (see Fig.3.1) are traction free. Solution (3.1) is one term in the decomposition mentioned in Section 2.

As usual, we introduce a mesh to partition Ω. For simplicity, we first consider the case of a uniform partition, characterized by the parameter h (see Fig.3.2).
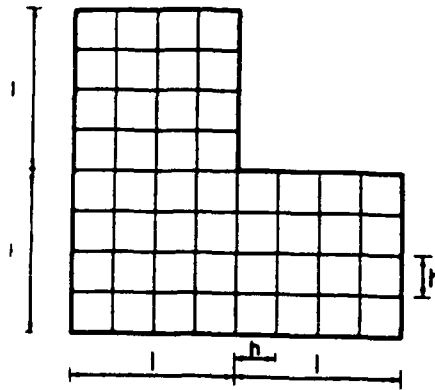
Fig.3.2 The uniform mesh.

The finite element spaces $V \subset H^1(\Omega)$ will consist of continuous piecewise polynomials of degree p on the squares of the mesh. The exact set of polynomials used over each square consists of either Q_p, the set of polynomials of degree p separately in each variable, or Q_p', the minimal set containing polynomials of total degree p. See eg. [26] for details. The finite element solution $\underline{u}_{FE}$ is then defined as usual by

$$B(\underline{u}_{FE}, \underline{v}) = F(\underline{v}) \quad \forall \underline{v} \in V$$

with the error satisfying

$$\|\underline{u}_0 - \underline{u}_{FE}\|_E = \min_{\underline{w} \in V} \|\underline{u}_0 - \underline{w}\|_E.$$

The space V (and hence also $\underline{u}_{FE}$) is characterized by two parameters, p and h, so that $V = V(p,h)$. By $N(p,h)$ we denote the dimension (i.e. the number of degrees of freedom) of $V(p,h)$. In order to obtain a desired accuracy for our approximation, we use an extension procedure, i.e. a procedure to increase the dimension $N(p,h)$. This can be of three types:

a) h-version. Here p is fixed, usually at a low level (eg. 1 or 2) and we achieve the desired accuracy by taking $h \rightarrow 0$.

b) p-version. In this case h is fixed, i.e. the same mesh is used and $p \rightarrow \infty$, i.e. the accuracy is achieved by increasing p.

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c) <u>h-p version</u>. Here h and p are simultaneously changed (either uniformly or selectively).

In general, we will be interested in the relative error $\|e\|_{ER} =$ $\|\underline{u}_0 - \underline{u}_{FE}\|_E / \|\underline{u}_0\|_E$. We present some theorems for our approximation which are the special cases of those proven in [27], [28].

Theorem 3.1 [27]

(3.2)
$$\|\underline{u}_0 - \underline{u}_{FE}\|_E \leq Ch^{\min(\lambda, p)} p^{-2\lambda}$$

where C is a constant independent of h and p.

The above theorem holds for both choices Q_p and Q'_p. In 2 dimensions, $N \approx h^{-2} p^2$ so that to obtain the optimal asymptotic rate minimizing N(p,h), we choose h = 1, i.e. the p-version. (Of course, N does not completely measure the needed work. Moreover, the accuracy measured in the energy norm is not necessarily the accuracy we are seeking in practice). Figs. 3.3a and 3.3b show the errors for the h- and p- versions respectively with elements of Q'_p type.
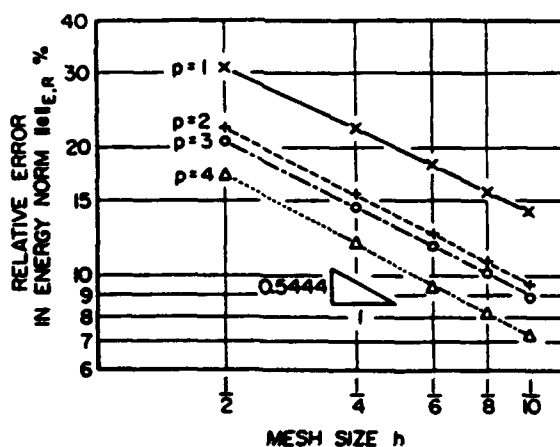
Fig. 3.3a The error for the h-version

Fig.3.3b The error for the p-version

The figures are drawn in the log log scale. They show that (3.2) correctly
characterizes the asymptotic behavior which is defined by the slope in the
figure.

Let us now consider a non-uniform geometric mesh with n layers and with
ratio 0.15. This is shown in Fig.3.4 for n = 2.

Fig.3.4 The geometric mesh

The shape functions are now the usual mapped polynomials using blending
mapping techniques for the circular sides. For details, see eg. [29].

In Fig.3.5 we show the error for different numbers of layers and the
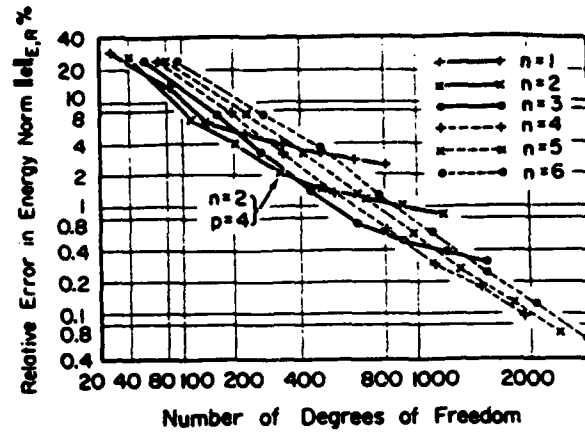(uniform) degree p in the log log scale

Fig.3.5 The error $\|e\|_{ER}$ as a function of n and p.

Fig.3.6 shows the error behavior in log $\|e\|_{ER} \times N^{1/3}$ for selected combinations of n and p.
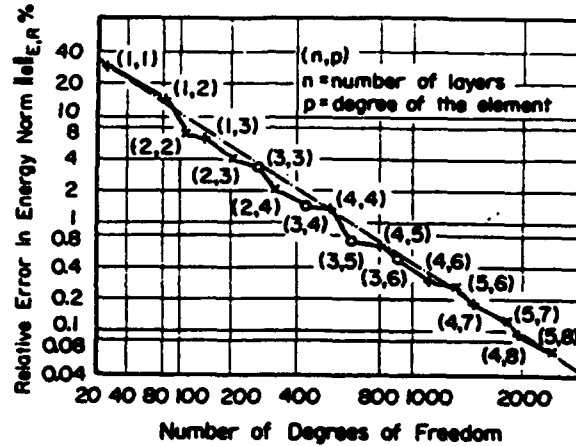
Fig.3.6 The error $\|e\|_{ER}$ for selected combinations of (n,p).

We see that $\|e\|_{ER} \approx Ce^{-\gamma^3\sqrt{N}}$. Estimates of this type have been analyzed in [30], [31], [32], employing the smoothness characterization of (2.9). We have

Theorem 3.2 [32]. Let the mesh with n layers be considered and let

$$\mu n \leq p \leq \nu n, \quad p > 1 \quad 0 < \mu < \nu < \infty.$$

Then if u_0 satisfies (2.9), we have

$$\|\underline{u}_0 - \underline{u}_{FE}\|_E \leq Ce^{-\gamma^3\sqrt{N}} \ , \quad \gamma > 0$$

In actual computations, the remeshing required by the h-p version is

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a disadvantage due to increased human cost. Consequently, the usual practice while employing p and h-p versions codes is to use a fixed, strongly refined mesh and then increase p (i.e. the p-version). The mesh design should be such that the desired accuracy will, as far as possible, be achieved for the optimal pair (n,p). In [33], [34] an attempt to design an expert system for such selection is presented.

Theorem 3.1 is a special case of the following, more general theorem, proved in [27].

Theorem 3.3 [27]   Assume that $\underline{u}_0$ has the form (2.8) with $\alpha = \min \alpha_j$. Then

$$\|\underline{u}_0 - \underline{u}_{FE}\|_E \le Cg(h,p,S) \ \min \left[h^\alpha, \frac{h^{\min(\alpha, p-\alpha)}}{p^{2\alpha}} \right], \ g(h,p,S) = \max \left(|\log h|^S, |\log p|^S \right)$$

Theorems 3.1 and 3.3 show the interesting fact that the convergence rate of the p-version is twice that of the h-version when a uniform or (more generally) a quasiuniform mesh is used. This fact was proved in [11] for the p-version (see also [35]).

In this connection, the following result from [27] is useful when $\underline{u}_0$ is only known to be in some Sobolev space as in (2.10).

Theorem 3.4 [27].   Let $\underline{u}_0 \in H^k(\Omega)$, $k > 1$. Then if the spaces $V = V(p,h)$ are based on a uniform (or quasiuniform) family of meshes,

(3.3)
$$\|\underline{u}_0 - \underline{u}_{FE}\|_E \le Ch^\mu p^{-(k-1)} \|\underline{u}_0\|_{H^k(\Omega)}$$

where $\mu = \min (p,k-1)$ and C is independent of $\underline{u}_0, h, p$.

Theorem 3.4 improves the classical estimate

$$\|\underline{u}_0 - \underline{u}_{FE}\|_E \le C(p)h^\mu \|\underline{u}_0\|_{H^k(\Omega)}$$

for the h-version by explicitly showing how the constant C(p) decreases when p is increased. Note that theorem 3.3 is a more refined result for solutions

of the form (2.8), since using (2.10) with theorem 3.4 will not yield the observed doubling in the rate of convergence of the p-version.

The p-version has been analyzed for 3-dimensional problems in [36], [37].

Various problems arising, for example, from the theory of plates and shells may be described by elliptic equations of order 2m where m > 1. For such problems, if the elements used are conforming piecewise polynomials, then they must have m-1 continuous derivatives over $\bar{\Omega}$. Approximation results for the p-version using such C^{m-1} elements have been established in [38], where it is shown that once again, due to the presence of r^α type singularities in the solution, one obtains twice the rate of convergence of the h-version. The case m=2 was originally discussed in [39] where some computational results using C^1 elements are presented. Theorems for the h-p version for equations of order 2m are given in [40].

For second order problems, the results in theorems 3.1-3.4 hold not only for square or triangular elements but also for curvilinear elements having some uniformity properties with respect to their mapping onto standard elements. For details, see eg. [31].

As we have seen, the presence cf singularities significantly decreases the rate of convergence. In addition, the p-version is influenced by the "pollution" problem [41]. By this we mean the effect of an error in one element (usually due to a singularity present in the true solution over that element) permeating into adjacent elements (where the exact solution is regular). The pollution problem is more serious if the stresses are of interest. It may be overcome by using refined meshes (a few layers) in the area of singularity. Another approach to deal with this problem is to use properly mapped shape functions. For details, see [42].

So far, we have assumed in our model problem that the Dirichlet boundary set is empty. The theorems we mentioned above are valid without changes when

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$\omega = 0$ on D i.e. the Dirichlet conditions are homogeneous. Then we simply use $^0V(p,h) = V(p,h) \cap {}^0H^1(\Omega)$ instead of $V(p,h)$.

In the case of nonhomogenous essential boundary conditions, we have to approximate ω by ω_{FE} so that ω_{FE} is in the trace space of the finite element space $V(p,h)$. This is done by a projection in the $H^\gamma(\Gamma_j)$ norm, $0 \leq \gamma \leq 1$. The cases $\gamma = 1, 1/2$ have been analyzed in [28], [43] and the general case in [44]. The results show that for smooth ω, the optimal rate of convergence is achieved when $1/2 \leq \gamma \leq 1$. More precisely, Theorem 3.4 holds for $1/2 \leq \gamma \leq 1$ and $\omega \in H^s(\Gamma_j)$, $s > 1$. For ω unsmooth, e.g. $\omega \in H^s(\Gamma_j)$, $\frac{1}{2} < s < \frac{3}{4}$, the optimal rate of convergence using the projection approach has been established only for $\gamma = \frac{1}{2}$. Numerical results are given in [30] and [44]. This problem does not occur with the h-p version when ω is singular in the neighborhood of the vertices. In [45], we have analyzed a class of constrained boundary conditions which are important in practice in structural mechanics.

So far we have only mentioned the solution of elliptic problems. The p and h-p version can also be used for other types of problems. For example, in [46], [47] we analyze the method for solving parabolic equations when the h-p version is used in both the time and space variables.

4. The problem of optimal meshes and adaptive approaches.

In the previous section, Fig. 3.3a showed the convergence rate for the h-version using a uniform mesh. This rate may be improved by using better meshes in certain cases.

The problem of optimal meshes for the h- and h-p versions was studied in detail for 1 dimension in [48] and for 2 dimensions in [49]. Let us mention some one dimensional results.

We consider the simple model problem

$$-u'' = f, \quad u(0) = u(1) = 0$$

with the exact solution $u(x) = x^\alpha - x$, $\alpha > \frac{1}{2}$. Let x_i denote the mesh points. For the h-version, the radical mesh $x_i = \left[\frac{i}{m}\right]^\beta$ $i = 1, \cdots, m$ is optimal.

Theorem 4.1 [48]. The radical mesh with $\beta = \dfrac{p + \frac{1}{2}}{\alpha - \frac{1}{2}}$ is optimal and

$$\lim_{m \to \infty} m^p \|u - u_{FE}\|_E = C(\alpha, p) \frac{\left[p + \frac{1}{2}\right]^{p + \frac{1}{2}}}{\alpha - \frac{1}{2}}$$

where

$$C(\alpha, p) = \frac{\alpha \Gamma(\alpha) |\sin \pi \alpha| \ \Gamma(p - \alpha + 1)}{\sqrt{\pi} \ 4^p \sqrt{2p+1} \ \Gamma(p + 1/2)} \ .$$

Theorem 4.1 shows that for the h-version, the best possible rate of convergence is $O(h^p)$, which is <u>algebraic</u>, and not <u>exponential</u> (as for the h-p version).

For the h-p version, the optimal mesh is a geometric one with ratio q, $x_i = q^{m-i}$, $0 < q < 1$, $i = 1, 2, \cdots, m$ and the optimal p-distribution is linear, $p_i = [si] + 1$ where $[a]$ denotes the integral part of a and p_i is the degree of the element (x_{i-1}, x_i). s will be called the <u>slope</u>. In the case of uniform p, we use $p = [sm] + 1$. The optimality is understood in the sense that the error using the optimal mesh and optimal degree distribution has the same exponential rate $q_{opt}^{\sqrt{(\alpha-1/2)N}}$ as the best achievable rate among all mesh and p distributions with the same number of degrees of freedom N.

Theorem 4.2. We have $q_{opt} = (\sqrt{2}-1)^2 \approx 0.17$ and $s_{opt} = 2\alpha - 1$. Then

$$\|u_0 - u_{FE}\|_E \le C(\alpha) q_{opt}^{\sqrt{(\alpha-1/2)N}}$$

In the case of uniform degree distribution we have

Theorem 4.3. $q_{opt} = (\sqrt{2}-1)^2$, $s_{opt} = 2\alpha-1$ and

$$\|u_0 - u_{FE}\|_E \le C(\alpha)q_{opt}^{\sqrt{(\alpha-1/2)N}} N^{-\sigma/2}$$

$$\sigma = \min (2\alpha - 1, \alpha).$$

We have seen that for p uniform, the radical mesh is optimal. Hence we can ask about the envelope of optimal radical meshes. Then the radical meshes tend to a geometric one with ratio $q = e^{-4/e^2} \approx 0.54$ and $s = 4(\alpha-1/2)/e^2 \approx 0.54(\alpha-1/2)$. These, together with many more detailed results as well as numerical experimentation are given in [48].

In two dimensions, the situation is more complicated. Nevertheless, the linear distribution of p and geometric mesh are once again optimal. The estimates will be of the type $e^{-\gamma^3\sqrt{N}}$ in contrast to $e^{-\gamma\sqrt{N}}$ in one dimension. For circular elements and optimal choice of the degrees of elements in different directions, we can achieve the rate $e^{-\gamma\sqrt{N}}$ too. For details we refer to [49].

The above results indicate that the geometric mesh with $q \approx 0.15$ is the right mesh for practical use. It is preferable to over-refine the mesh slightly. The selection of the number of layers can be made in an expert system mode or adaptively. Adaptive approaches were addressed in [49], [50]. Let us mention that the codes FIESTA and STRIPE have some adaptive features with shape functions being selected in an anisotropic way. In PROBE, the determination of the p-distribution is done at present by the user.

5. The p and h-p version for integral equations and mixed methods.

The theorems in Section 3 (and 4) were based on approximation theory results in the H^1 norm. We can proceed analogously for cases where we have a

coercive bilinear form over some other space H for which corresponding regularity and p- and h-p version approximation results are known. In [51] this procedure is extended to the boundary element method. Consider for example the model problem from Section 2, of Laplace's equation on a polygonal domain when both the Dirichlet and Neumann sets are present. Then the problem can be formulated on Γ^Ω (see [51]) as a system of integral equations with the unknowns being given by the pairs $\left[\Phi|_{\Gamma_2}, \frac{\partial \Phi}{\partial n}|_{\Gamma_1} \right]$ where $\Gamma_2 \in \mathcal{N}$, $\Gamma_1 \in D$. This problem may be put in the form $B(u,v) = F(v)$. It satisfies a Garding inequality which is sufficient to obtain an optimal rate of convergence for the Galerkin approximation. In [52], the h and h-p versions were analyzed and it was shown that the rate of convergence of the p-version is twice as high as that for the h-version (with uniform mesh), similar to the cases discussed earlier. We can also obtain an exponential rate of convergence for the h-p version with a properly chosen (geometric) mesh and degrees analogously selected as in the previous sections. For details, see [53], [54], [55]. For adaptive procedures in the h-p version for integral equations, we refer to [56].

The problems discussed so far have been <u>stable</u> (in the sense that they are coercive or a Garding inequality holds). For mixed methods, one must first establish the stability of the approximate subspaces used, via an inf-sup condition. The stability of the p-version in the context of certain mixed methods for Stokes' problem has been discussed in [57], [58]. (See also spectral method references). In [59], the Raviart-Thomas and the Brezzi-Douglas-Marini spaces for the mixed formulation of linear elliptic problems have been shown to be stable and possess optimal convergence properties in terms of the h-p extension using quasiuniform meshes. The p-extension of Raviart-Thomas elements for quasilinear problems is analyzed in [60].

6. The h-p version and mathematical modeling.

Let us consider as a model problem the problem of plate bending. This problem is, in fact, a 3 dimensional problem over a "thin" domain $\Omega = \omega \times (-t/2, t/2) \subset \mathbb{R}^3$. Two dimensional formulations such as Kirchhoff or Reissner-Mindlin models (among others) are dimensionally reduced formulations of this problem. These formulations generally are asymptotically identical for $t \rightarrow 0$, but yield different results for $t > 0$. The solutions of these 2 dimensional models have to be understood as approximations of the 3 dimensional formulation. The error depends on the type of input data (for eg., clamped or simply supported plate), thickness and the aim of the computation. The h-p version gives a natural tool which leads to a hierarchical set of formulations. Denoting the displacements by $\underline{u} = (u_1, u_2, u_3)$, the dimensional reduction can be understood as a projection on the space of solutions of the form

$$u_1(x_1, x_2, x_3) = \sum_{j=1}^{s_1} u_1^{(j)}(x_1, x_2) x_3^{2j-1}$$

(6.1)
$$u_2(x_1, x_2, x_3) = \sum_{j=1}^{s_2} u_2^{(j)}(x_1, x_2) x_3^{2j-1}$$

$$u_3(x_1, x_2, x_3) = \sum_{j=0}^{s_3} u_3^{(j)}(x_1, x_2) x_3^{2j} .$$

For example if $\nu = 0$ (ν = Poisson ratio) then the choice $s_1 = s_2 = 1$, $s_3 = 0$ leads to the Reissner-Mindlin model. For $\nu > 0$ one has to take $s_3 = 1$. For more about dimensional reduction we refer to [61] and references therein. The error of the reduced formulations depends on various factors. For example (see [62]), for the simply (soft) supported uniformly loaded Reissner-Mindlin plate with angle $30°$, side length 1 and thickness $t = 0 \cdot 1$, $0 \cdot 01$, the errors

in the energy norm are 4.34% and 15.41% respectively. The form (6.1) can be understood as the p-version with the polynomial degrees in the x_3 direction being different from those in x_1, x_2. From this point of view, the h-p version is a natural tool for deriving plate models and assessing their error (see next section). The program PROBE has these types of features for application to plates and shells, as well as for transitions where s is changed in various parts of the domain. As shown in [62] and [63], the various boundary conditions (hard, soft) have a significant influence on the solution. For more, we also refer to [64].

7. <u>Extraction techniques.</u>

Usually in computational practice, the solution u of our variational problem is only a tool to get the primary quantity of interest. For example, the goal of the computation may be to find the stresses at a point, or the maximal stress (e.g. Mises equivalent stress) over a region, or the resultants (reactions, moments) in the plates and shells, stress intensity factors, etc. Mathematically, we are interested in evaluating the values of certain functionals. This can be done in a trivial way (for example, by differentiating the finite element solution u_{FE}) or using more sophisticated approaches which lead to more accurate results (with accuracy being of the order of the error in the energy, rather than the energy norm). Such techniques, called extraction techniques, were addressed, for e.g., in [65] and applied in various important contexts (see [66], [67]).

As an example of an extraction technique, we present computations for the stress intensity factors for the model problem introduced in Section 3, but with the exact solution consisting of two terms of the expansion, $\lambda_1 = 0.54448$ and $\lambda_2 = 0.90853$. We have selected the intensity factors $\alpha_1 = 1$ (mode 1)

and $\alpha_2 = 2$ (mode 2). Fig.7.1 shows the error of α_1 and α_2 in the scale $\log e \times N^{1/3}$ for the two layer mesh as well as the error in the strain energy (not energy norm).
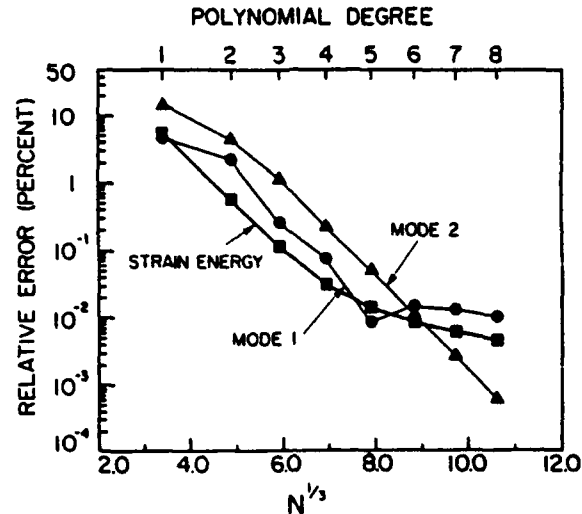

Fig.7.1 Convergence of the stress intensity factors computed by the extraction technique.

We see that, in fact, the accuracy in the stress intensity factors is of the same order as that in the strain energy and that for high p, the second mode is more accurate than the strain energy (see [65] for an analysis). As we see, the error does not behave monotonically. (The computations above were performed by PROBE, which offers this extraction technique feature). An essential prerequisite of the extraction here is the knowledge of the coefficients α_j and $\psi_j(\theta)$ in (2.8) (and the adjoint of $\psi_j(\theta)$). As shown in Section 2, these are available. For the extraction of other data of interest, we refer, for e.g., to [67]. Although they are not computationally trivial, extraction techniques can potentially save a large amount of computer time (when included as a standard feature in a program), especially in 3 dimensions, where an error of order 1% in stain _energy_ is easy to obtain but an error of order 1% in the energy _norm_ is very difficult to achieve.

8. A-posteriori estimates.

An essential aspect of the finite element method is quality control of the computed results of interest. The p-version (on properly designed meshes) gives an effective tool for this, because it computes extensions without changing the mesh (and saves precious user time). The computed sequence of the value of interest can be analyzed by various extrapolation approaches (or simply by assessing the changes with p visually). If the data is monotonic, as in the case of the energy, the extrapolation technique is very effective.

Table 8.1 shows the approximate relative energy norm error estimates by the program PROBE for the model problem mentioned in Section 3, with $n = 2$ layers.

Table 8.1 The estimated and true energy norm errors.

p	N	Estimated Error	True Error	p	N	Estimated Error	True Error
1	41	25.41	25.41	5	497	1.41	1.47
2	119	8.45	8.45	6	695	1.15	1.32
3	209	3.91	3.93	7	729	0.89	0.98
4	335	2.09	2.13	8	1199	0.74	0.85

The estimates are computed by using extrapolation based on the formula

$$(8.1) \qquad \|e\|_E^2 = CN^{-\beta} = E_{EX} - E_{FE}$$

where E_{EX} (respectively E_{FE}) is the exact (respectively computed) finite element energy. (Note that there are 3 unknowns in (8.1): C, β, E_{EX}.). We compute E_{EX} out of three successive values. The final value of E_{EX} accepted

is then used for all p. When the error curve is concave (as we would like to achieve by the proper mesh and an exponential rate of convergence), then the estimated errors are upper bounds. In any case, as Table 8.1 shows, the error estimate is of high quality. We can and should use other quality controls, e.g. various equilibrium checks, etc. (see e.g. [68]). PROBE has various such features. Essentially, one has to compute the values of interest more accurately than needed for engineering purposes because of quality control reasons.

If the values are not monotonic than it is easiest to present the entire sequence to the user. As an example, we show in Fig.8.1 the 3 dimensional analysis of a splice and depict the maximal principle stress in the region. The standard h-version computation results are also given. The data are taken from [69].

It is observed that one may decide by inspection that the p-version has converged satisfactorily. A similar deduction is not possible with the h-version.
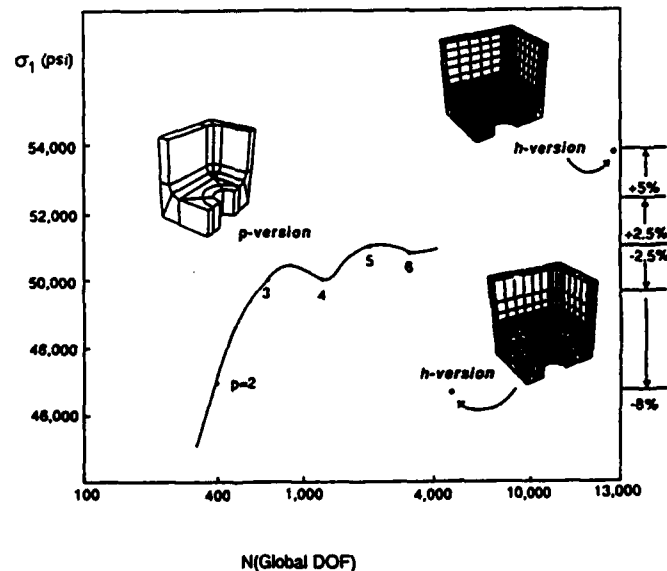

Fig. 8.1 The accuracy of the maximal principal stresses in a splice computation

Similar principles can be used in the error assessment of the modeling mentioned in Section 6 and extraction techniques mentioned in Section 7.

9. Robustness and problems involving "locking".

A robust method is one which performs uniformly well for a broad class of input data. Consider for example the elasti~.~y problem as defined in Section 2. When $\nu \rightarrow 1/2$, the form degenerates (although the solution stays smooth, see Section 2) and we have $\operatorname{div} u \rightarrow 0$. It is well known that the h-version of the finite element method for low degree p performs very poorly, due to the phenomenon called "locking".

There are various types of locking. The above mentioned is called Poisson locking and will be briefly discussed here. Essentially, when $\nu \rightarrow 1/2$, problems arise in a) the convergence in the energy norm and b) computation of the pressure $(\sigma_x + \sigma_y)$.

We have to distinguish between the case of straight and curved elements. It has been shown in [70] that the rate of convergence in the energy norm of the p-version with straight triangles is not influenced by $\nu \rightarrow 1/2$. In [71], it has been shown that the h-version with straight triangles does not show locking when $p \geq 4$. The general case is analyzed in [72], [73]. Theorems 9.1 and 9.2 are specialized versions of the more general results obtained in these references.

Theorem 9.1 (3.3) holds for straight-sided triangles and parallelograms uniformly in ν provided that $p \geq p_0$ where $p_0 = 4$ for triangles, 3 for Q_p elements and 5 for Q'_p elements.

The above theorem shows that with straight triangles and parallelograms, no locking occurs for $p \geq p_0$ both with the h and the p-version. For curvilinear elements, we have for the p-version:

<u>Theorem 9.2</u>. Let $u_0 \in H^k(\Omega)$, $k > 1$. Then the following estimate holds for the p-version uniformly in ν,

$$\|u_0 - u_{FE}\|_E \leq C(p-s)^{k-1} \|u_0\|_{H^k(\Omega)}$$

where $s \geq 0$ depends upon the mappings of the curvilinear elements onto the standard elements, provided these mappings are rational.

A related theorem for the case when the mappings are analytic may be found in [72], [73].

As a simple illustration, we show the relative error in the energy norm for various straight and curvilinear choices of a single element for $\nu = 0.3$ and $\nu = 0.5 - 10^{-10}$, where a Q'_p type element is used. For a detailed analysis and numerical examples, we refer to [72], [73].
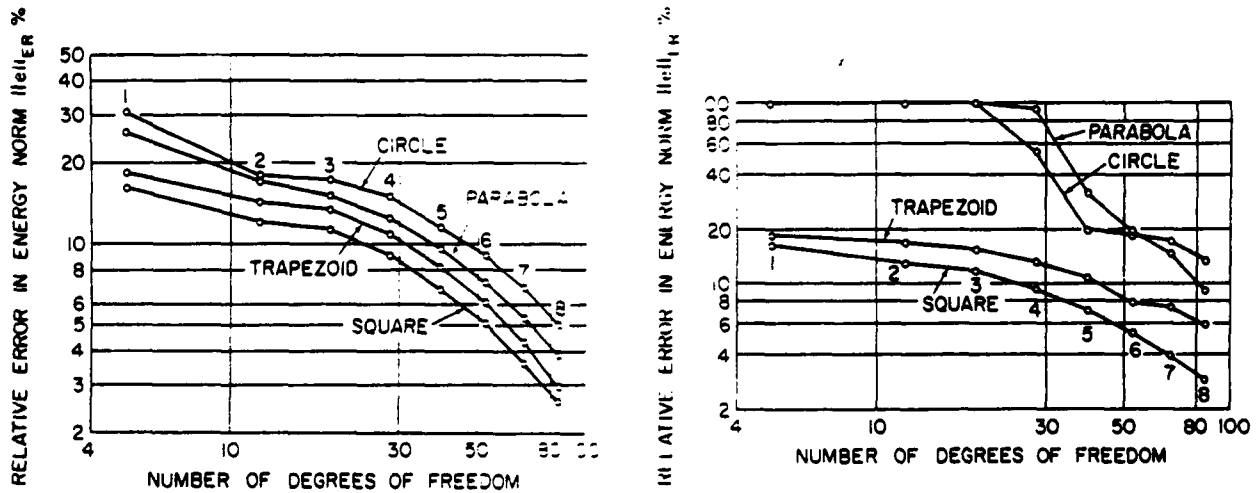
Fig. 9.1 Error behavior for curved elements for $\nu = 0.3$ and $\nu = 0.5-10^{-10}$.

The second problem is the pressure recovery. In [74], [75], it is shown that the stress components $\sigma_x - \sigma_y$, τ_{xy} are accurately computable directly from the solution (by Hooke's Law) but the directly computed pressure is unusable. Nevertheless, the pressure can be accurately recovered by a post-processing technique based on the observation that it is a harmonic function and that u_{FE} is accurate in the energy norm.

A similar behavior occurs for other cases of material which can be associated with nearly degenerate bilinear forms (see [14]) in two and three dimensions. The h-p version is also very robust in relation to the shear locking that occurs in plate theory.

10. Implementational aspects.

In contrast to the h-version, the p-version needs much more computational work to construct the local stiffness matrices and load vectors. Moreover, it leads to matrices which are less sparse. On the other hand, the local stiffness matrix computation is completely parallel (and for uniform p is well balanced) which can obviously be implemented by parallel computers.

For complex geometries, curvilinear elements with relatively large distortion cannot be avoided. This problem is overcome in the p-version by using quadrature rules with the number of quadrature points depending on the distortion.

The system of equations for the FEM solution is less sparse for high p than for low p. Hence the solution is more expensive for high p. Nevertheless, the ratio of the computational work to the accuracy obtained is more favorable for the p-version (this is also true for engineering accuracy). For a detailed analysis we refer to [76], [77].

Iterative method techniques like the conjugate gradient method can be very favorably influenced by the correct selection of the shape functions. For various aspects of the influence of the shape functions on the iterative process, we refer to [78]. In [79] we have shown that in 2 dimensions, the preconditioned conjugate gradient method (preconditioning by $p = 1$ computation) requires $O(\log p)$ steps asymptotically when the shape functions are properly chosen.

In [80], detailed experimentation and analysis of the factors influencing effectiveness and parallelization on Alliant computers are presented. It is shown that the speed-up is at least 90%.

In general, mesh generation, especially in 3 dimensions, is a difficult task. Presently, a mesh generator geared to the needs of the p-version is not available and a PATRAN interface is usually used.

For experimentation with mesh refinement of the h-p version on tensor product meshes in two dimensions, we refer to [81].

11. Engineering experience and practice.

In the previous section, we discussed various theoretical aspects of the h-p version. A large amount of engineering and industrial experience with the method has been gained in connection with the use of commercial programs FIESTA, PROBE and research program STRIPE. For some articles, we refer to [82] and references therein. See also [83].

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